B. Sc. Part III (Hons).

7^{th} Paper (Diff. Eqns)

Q. Solve $x^2 \frac{d^2y}{dx^2} - (x^2+2x)\frac{dy}{dx} + (x+2)y = x^3 e^x$

Soln The given equation

$$\frac{d^2y}{dx^2} - \frac{x^2+2x}{x^2}\frac{dy}{dx} + \frac{x+2}{x^2}y = xe^x \quad \text{---} (1)$$

It is of the form

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R.$$

Here,

$$\left.\begin{aligned} P &= -\frac{x^2+2x}{x^2} = -\left(1+\frac{2}{x}\right) \\ Q &= \frac{x+2}{x^2} = \frac{1}{x}+\frac{2}{x^2} \\ R &= xe^x \end{aligned}\right\} (2)$$

Now, $P + Qx = -(\cancel{1}+\cancel{\tfrac{2}{x}}) + x\left(\cancel{\tfrac{1}{x}}+\cancel{\tfrac{2}{x^2}}\right) = 0$

$\Rightarrow$ $u = x$ is a part of the eq(1).

Let $y = uv$ —— (3) be the general soln.

$\Rightarrow$ v is given by

$$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u}$$

$$\Rightarrow \frac{d^2v}{dx^2} + \left[-\left(1+\frac{2}{x}\right) + \frac{2}{x}\cdot\frac{dx}{dx}\right]\frac{dv}{dx} = \frac{xe^x}{x}$$

$$\Rightarrow \frac{d^2v}{dx^2} + \left[-\left(1+\cancel{\frac{2}{x}}\right) + \cancel{\frac{2}{x}}\right]\frac{dv}{dx} = e^x$$

$$\Rightarrow \frac{d^2v}{dx^2} - \frac{dv}{dx} = e^x \qquad \text{Put } \frac{dv}{dx} = z$$

$$\Rightarrow \frac{dz}{dx} - z = e^x$$ which is a linear eqn.

$$IF = e^{-\int dx} = e^{-x}$$

Its soln is $\quad z \cdot IF = \int e^x \cdot IF \, dx$

$$\Rightarrow z \cdot e^{-x} = \int e^x \cdot e^{-x} \, dx$$

$$\Rightarrow z e^{-x} = x + c_1$$

$$\Rightarrow z = xe^x + c_1 e^x \qquad \text{But } z = \frac{dv}{dx}$$

$$\Rightarrow \frac{dv}{dx} = xe^x + c_1 e^x$$

$$\Rightarrow dv = (xe^x + c_1 e^x)\,dx$$. integrating, we get

$$\Rightarrow v = \int xe^x \, dx + c_1 \int e^x \, dx$$

$$\Rightarrow v = x\int e^x \, dx - \int\left[\frac{d(x)}{dx}\int e^x \, dx\right]dx + c_1 e^x$$

$$\Rightarrow v = xe^x - e^x + c_1 e^x + c_2$$

$$\Rightarrow v = (x - 1 + c_1)e^x + c_2$$

Hence $y = uv$ is the soln where $u = x$ and

$$v = (x - 1 + c_1)e^x + c_2$$

$$\Rightarrow y = x(x - 1 + c_1)e^x + c_2 x$$ is the complete soln.